M.Math. IInd year
Second semestral exam 2013
Algebraic Number Theory
Instructor — B.Sury
ANSWER ANY FIVE

Q 1.
Prove that the class group of $(\sqrt{-10})$ has order 2.

OR

Let K be a number field of degree 3 over $\mathbf{Q}$ such that $disc(K) = -31$. Show that O_K is a PID.

Q 2.
Consider a cyclotomic field $\mathbf{Q}(\zeta)$ where ζ is a primitive (2^n+1)-th root of unity. If $\Phi(X)$ denotes the minimal polynomial of ζ over $\mathbf{Q}$, consider the reduction $\overline{\Phi}(X)$ of $\Phi(X)$ mod 2 in $(\mathbf{Z}/2\mathbf{Z})[X]$. Determine the number of irreducible factors of the polynomial $\overline{\Phi}(X)$ and their degrees.

Q 3.
Let k be complete with respect to a nonarchimedean valuation $|.|$. Let l be a finite extension of k. Prove that $|x|_l := |N_{l/k}(x)|^{1/[l:k]}$ is a nonarchimedean valuation on l.

OR

Prove that $\mathbf{Q}_p$ contains $p-1$ distinct roots of unity.

Q 4.
Let N/K be a normal extension of number fields, with Galois group G. For a subgroup H of G, consider $L = N^H$. Let $G = \bigsqcup_{i=1}^n Hg_i$. View each element x of G as a permutation of $\{1, 2, \cdots, n\}$ (by $Hg_ix = Hg_{x(i)}$). Let x be the Frobenius automorphism of an unramified prime ideal P of O_N. If x is a product of cycles of lengths $f_1, \cdots, f_k$, prove that the ideal $(P \cap O_K)O_L$ is a product of k distinct prime ideals, which have the residue degrees $f_1, f_2, \cdots, f_k$.

OR

For any modulus $\mathbf{m}$ of a number field K, define the groups $K_{\mathbf{m}}$ and $K_{\mathbf{m},1}$. Further, show that the quotient $K_{\mathbf{m}}/K_{\mathbf{m},1}$ is finite.

Q 5.
Determine the rank and torsion group of units of the ring of integers of $\mathbf{Q}(\zeta_{p^n})$ where p is an odd prime.

OR

Let α be a root of $f(X) = X^5 - X - 1$ and let $K = \mathbf{Q}(\alpha)$. The polynomial f is irreducible and the discriminant of K is 19×151 - assume these. Show that the ideal $(19, \alpha + 6)^2$ divides $19O_K$.

Q 6.
Consider $L := \mathbf{Q}(\zeta_{p^2})$, where ζ_{p^2} is a primitive p^2-th root of unity with p an odd prime. Let K be the unique subfield of L which has degree p over $\mathbf{Q}$. Show that 2 splits completely in K if and only if $2^{p-1} \equiv 1 \bmod p^2$.

OR

Consider $L := \mathbf{Q}(\zeta_{31})$, where ζ_{31} is a primitive 31-st root of unity. Let K be a subfield of L which has degree 3 over $\mathbf{Q}$. Prove that $O_K \neq \mathbf{Z}[\alpha]$ for any α.

Q 7.
Define the discriminant of a number field K. Determine with proof the sign of disc K. Further, describe the embedding of O_K as a lattice in $\mathbf{R}^n$ where $[K : \mathbf{Q}] = n$ and determine its volume.

OR

For the number field $K = \mathbf{Q}(\alpha)$ where $\alpha^5 = 2\alpha + 2$, prove that $O_K = \mathbf{Z}[\alpha]$.

Q 8.
Prove that $(1 - \zeta_p)^{p-2}$ is the different ideal of $\mathbf{Q}(\zeta_p)$ where p is a prime, and ζ_p is a primitive p-th root of unity.

OR

State the Frobenius density theorem. Use it to deduce that the density of primes which split completely in a Galois extension is the reciprocal of the order of the Galois group.